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On the Value of Annuities payable Half-yearly, Quarterly, &c.

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IN the last Number of this *Journal* I examined and compared the solutions of this question given by Milne, Baily, and Griffith Davies in their standard works. The first of these writers cannot be considered to have treated the question in a satisfactory manner. The two latter have investigated the problem more rigorously. They both proceed upon the supposition commonly adopted, viz. that the deaths occurring in each year of age are distributed uniformly over the year. They differ, however, in their views as to the interest to be allowed for fractions of a year, Davies throughout adopting Milne's view, by which the present value of £1 due in six months' time is $\dfrac{1}{1+\frac{i}{2}}$, while Baily takes the value as $\dfrac{1}{\sqrt{1+i}}$. The latter, as I have already stated, appears to me the preferable view. Davies, however, has also given an investigation of the value of an annuity payable m times a year, on the supposition that interest is convertible into principal m times in the year; and his solution, as I have shown in the last Number of the *Journal*, is substantially the same as Baily's.

The only authors who have attempted, so far as I am aware, to carry the investigation of the question beyond the point at which

it has been left by the above writers, are Professor De Morgan, who gave, without demonstration, a formula in the *Companion to the Almanac* for 1842; and Mr. Woolhouse, who has quite recently given for the first time (*Assurance Magazine*, vol. xi. p. 327) a formula at once very simple and very accurate. The paper of the former is reprinted in the last Number of this *Journal*; and it will be seen upon reference that the formula there given is an exact one—*i.e.* not approximate merely—obtained on the supposition that the deaths in each year proceed by constant third differences, in a manner depending on the mortality of the preceding and following years.* It is, however, far too complicated to be of much use, and it is open to the further objection—a more serious one, in my opinion—that it is based on Milne's supposition as to the interest to be allowed for fractions of a year.

I will now, therefore, proceed to obtain the formula resulting from the adoption of De Morgan's supposition as to the mortality, combined with Baily's supposition as to the interest for fractions of a year. Thus, l_k denoting the number living at the age k, let the number living at the age $k+x$ (where x is less than 1) be

$$l_{k+x}=l_k+\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3 \quad . \quad . \quad . \quad . \quad (1)$$

Here making $x=-1, 1, 2$, successively, in conformity with our supposition that the deaths between the ages k and $k+1$ are to depend on the deaths of the preceding and the following years, we get

$$\begin{aligned} l_{k-1}&=l_k-\mathrm{A}+\mathrm{B}-\mathrm{C} \\ l_{k+1}&=l_k+\mathrm{A}+\mathrm{B}+\mathrm{C} \\ l_{k+2}&=l_k+2\mathrm{A}+4\mathrm{B}+8\mathrm{C}. \end{aligned}$$

From these equations we find

$$\begin{aligned} \mathrm{A}&=-\frac{1}{3}l_{k-1}-\frac{1}{2}l_k+l_{k+1}-\frac{1}{6}l_{k+2}, \\ \mathrm{B}&=\frac{1}{2}l_{k-1}-l_k+\frac{1}{2}l_{k+1}, \\ \mathrm{C}&=-\frac{1}{6}l_{k-1}+\frac{1}{2}l_k-\frac{1}{2}l_{k+1}+\frac{1}{6}l_{k+2}. \end{aligned}$$

Then, each year being divided into m equal parts, at the end of

* There is in Professor De Morgan's paper what appears to me, if not an error, at least an ambiguity, of language. He says (p. 138), "If *the mortality* of the table be increasing from year to year, it ought to be supposed that the mortality of the latter part of a year is greater than that of the former." The meaning of this is not, as might be supposed at first sight, "If the *rate of mortality* be increasing," &c.; but from the context it appears that the meaning intended is, "If the number dying in a year be increasing from year to year, it ought to be supposed that the number dying in the latter part of the year is greater than the number dying in the former part." I should be glad to learn whether the learned author, on reflection, considers that his words accurately express his meaning.

each of which a payment $\frac{1}{m}$ is to be made, the value of the payment so made at the end of the xth part of the $(t+1)$th year will be

$$\frac{1}{m}\cdot v^t\cdot\frac{l_{k+t+\frac{x}{m}}}{l_k}\cdot v^{\frac{x}{m}},$$

where the value of $l_{k+t+\frac{x}{m}}$ is to be found in terms of l_{k+t-1}, l_{k+t}, l_{k+t+1}, l_{k+t+2}, by means of the formula (1).

Hence the value of all the payments in the $(t+1)$th year

$$=\frac{v^t}{ml_k}\Sigma_x v^{\frac{x}{m}}\Big\{l_{k+t}+\frac{x}{m}\Big(-\frac{l_{k+t-1}}{3}-\frac{l_{k+t}}{2}+l_{k+t+1}-\frac{l_{k+t+2}}{6}\Big)$$
$$+\frac{x^2}{m^2}\Big(\frac{l_{k+t-1}}{2}-l_{k+t}+\frac{l_{k+t+1}}{2}\Big)+\frac{x^3}{m^3}\Big(-\frac{l_{k+t-1}}{6}+\frac{l_{k+t}}{2}-\frac{l_{k+t+1}}{2}+\frac{l_{k+t+2}}{6}\Big)\Big\}$$
$$=\frac{v^t}{ml_k}\Sigma_x v^{\frac{x}{m}}\Big\{l_{k+t-1}\Big(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\Big)+l_{k+t}\Big(1-\frac{x}{2m}-\frac{x^2}{m^2}+\frac{x^3}{2m^3}\Big)$$
$$+l_{k+t+1}\Big(\frac{x}{m}+\frac{x^2}{2m^2}-\frac{x^3}{2m^3}\Big)+l_{k+t+2}\Big(-\frac{x}{6m}+\frac{x^3}{6m^3}\Big)\Big\},$$

(x having the values $1, 2, 3 \ldots m$,)

$$=\frac{P}{m}v^t\cdot\frac{l_{k+t-1}}{l_k}+\frac{Q}{m}v^t\frac{l_{k+t}}{l_k}+\frac{R}{m}v^t\frac{l_{k+t+1}}{l_k}+\frac{S}{m}v^t\frac{l_{k+t+2}}{l_k},\ \text{suppose.}$$

And the total value of the annuity will be found by taking the sum of the values obtained by making t equal to $0, 1, 2 \ldots.$ to the extremity of life.

Now

$$\Sigma_t v^t\cdot\frac{l_{k+t-1}}{l_k}=\frac{l_{k-1}}{l_k}+v\cdot\frac{l_k}{l_k}+\frac{v^2l_{k+1}}{l_k}+\frac{v^3l_{k+2}}{l_k}+\&c.=\frac{l_{k-1}}{l_k}+v(1+a_k)$$

$$\Sigma_t v^t\cdot\frac{l_{k+t}}{l_k}=\frac{l_k}{l_k}+\frac{vl_{k+1}}{l_k}+\frac{v^2l_{k+2}}{l_k}+\ldots.=1+a_k,$$

$$\Sigma_t v^t\frac{l_{k+t+1}}{l_k}=\frac{l_{k+1}}{l_k}+\frac{vl_{k+2}}{l_k}+\ldots.=\frac{1}{v}\cdot a_k,$$

$$\Sigma_t v^t\frac{l_{k+t+2}}{l_k}=\frac{l_{k+2}}{l_k}+\frac{vl_{k+3}}{l_k}+\ldots.$$
$$=-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{1}{v^2}\Big(\frac{vl_{k+1}}{l_k}+\frac{v^2l_{k+2}}{l_k}+\frac{v^3l_{k+3}}{l_k}+\ldots\Big)=-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{a_k}{v^2}.$$

Therefore the value of the annuity is

$$\frac{P}{m}\Big\{\frac{l_{k-1}}{l_k}+v(1+a_k)\Big\}+\frac{Q}{m}(1+a_k)+\frac{R}{m}\cdot\frac{a_k}{v}+\frac{S}{m}\Big(-\frac{1}{v}\frac{l_{k+1}}{l_k}+\frac{a_k}{v^2}\Big)$$
$$=a_k\Big\{\frac{Pv}{m}+\frac{Q}{m}+\frac{R}{mv}+\frac{S}{mv^2}\Big\}+\frac{P}{m}\Big(\frac{l_{k-1}}{l_k}+v\Big)+\frac{Q}{m}-\frac{S}{mv}\frac{l_{k+1}}{l_k}\quad\ldots(2)$$

We have now to find the values of P, Q, R, S, as defined by the equations

$$P=\Sigma v^{\frac{x}{m}}\left(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\right),\quad Q=\Sigma v^{\frac{x}{m}}\left(1-\frac{x}{2m}-\frac{x^2}{m^2}+\frac{x^3}{2m^3}\right),$$

$$R=\Sigma v^{\frac{x}{m}}\left(\frac{x}{m}+\frac{x^2}{2m^2}-\frac{x^3}{2m^3}\right),\qquad S=\Sigma v^{\frac{x}{m}}\left(-\frac{x}{6m}+\frac{x^3}{6m^3}\right).$$

Put $v^{\frac{1}{m}}=c$; then the following formulæ may be proved for Σxc^x, Σx^2c^x, Σx^3c^x, the summation extending over values of $x=1, 2, 3 \ldots m$.

$$\Sigma(xc^x)=\frac{c-c^{m+1}}{(1-c)^2}-\frac{mc^{m+1}}{1-c},$$

$$\Sigma(x^2c^x)=\frac{(1+c)(c-c^{m+1})}{(1-c)^3}-\frac{2mc^{m+1}}{(1-c)^2}-\frac{m^2c^{m+1}}{1-c},$$

$$\Sigma(x^3c^x)=\frac{(1+4c+c^2)(c-c^{m+1})}{(1-c)^4}-\frac{3mc^{m+1}(1+c)}{(1-c)^3}-\frac{3m^2c^{m+1}}{(1-c)^2}-\frac{m^3c^{m+1}}{1-c}.$$

Then we have

$$P=\Sigma c^x\left(-\frac{x}{3m}+\frac{x^2}{2m^2}-\frac{x^3}{6m^3}\right)$$

$$=-\frac{1}{3m}\Sigma(xc^x)+\frac{1}{2m^2}\Sigma(x^2c^x)-\frac{1}{6m^3}\Sigma(x^3c^x).$$

Substituting from above the values of $\Sigma(xc^x)$, &c., we get, after reductions which present no difficulty,

$$P=-\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}+\frac{c(1+c)}{2m^2(1-c)^3}-\frac{2c+c^{m+1}}{6m(1-c)^2}.$$

Similarly

$$Q=\frac{(1+4c+c^2)(c-c^{m+1})}{2m^3(1-c)^4}-\frac{(1+c)(2c+c^{m+1})}{2m^2(1-c)^3}-\frac{c-2c^{m+1}}{2m(1-c)^2}+\frac{c}{1-c},$$

$$R=-\frac{(1+4c+c^2)(c-c^{m+1})}{2m^3(1-c)^4}+\frac{(1+c)(c+2c^{m+1})}{2m^2(1-c)^3}+\frac{2c-c^{m+1}}{2m(1-c)^2}-\frac{c^{m+1}}{1-c},$$

$$S=\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}-\frac{(1+c)c^{m+1}}{2m^2(1-c)^3}-\frac{c+2c^{m+1}}{6m(1-c)^2}.$$

Put $\frac{(1+4c+c^2)(c-c^{m+1})}{6m^3(1-c)^4}=H$; then, since $c^m=v$, we get

$$\begin{aligned}Pv^3+Qv^2+Rv+S= \;& v^3\left\{-H+\frac{c(1+c)}{2m^2(1-c)^3}-\frac{2c+cv}{6m(1-c)^2}\right\},\\ &+v^2\left\{3H-\frac{(1+c)(2c+cv)}{2m^2(1-c)^3}-\frac{c-2cv}{2m(1-o)^2}+\frac{c}{1-c}\right\}\\ &+v\left\{-3H+\frac{(1+c)(c+2cv)}{2m^2(1-c)^3}+\frac{2c-cv}{2m(1-c)^2}-\frac{cv}{1-c}\right\}\\ &+\quad H\quad-\frac{(1+c)cv}{2m^2(1-c)^3}-\frac{c+2cv}{6m(1-c)^2}\\ =\;&(1-v)^3H-\frac{c(1-v)^4}{6m(1-c)^2},\end{aligned}$$

and

$$\frac{Pv^3+Qv^2+Rv+S}{mv^2}=\frac{(1+4c+c^2)(c-cv)(1-v)^3}{6m^4(1-c)^4v^2}-\frac{c(1-v)^4}{6m^2v^2(1-c)^2}$$

$$=\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}.$$

Substituting in (2), the value of the annuity becomes

$$a_k\cdot\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}$$

$$+\left(\frac{l_{k-1}}{l_k}+v\right)\left\{-\frac{(1+4c+c^2)(c-cv)}{6m^4(1-c)^4}+\frac{c(1+c)}{2m^3(1-c)^3}-\frac{2c+cv}{6m^2(1-c)^2}\right\}$$

$$+\frac{(1+4c+c^2)(c-cv)}{2m^4(1-c)^4}-\frac{(1+c)(2c+cv)}{2m^3(1-c)^3}-\frac{c-2cv}{2m^2(1-c)^2}+\frac{c}{m(1-c)}$$

$$-\frac{l_{k+1}}{l_k}\left\{\frac{(1+4c+c^2)(c-cv)}{6m^4v(1-c)^4}-\frac{cv(1+c)}{2m^3v(1-c)^3}-\frac{c+2cv}{6m^2v(1-c)^2}\right\}$$

$$\left.\begin{aligned}&=\frac{c(1-v)^4}{6m^4v^2(1-c)^4}\{1+4c+c^2-m^2(1-c)^2\}\times a_k\\&-\frac{(1+4c+c^2)c(1-v)}{6m^4(1-c)^4}\left\{\frac{l_{k-1}}{l_k}+\frac{1}{v}\frac{l_{k+1}}{l_k}+v-3\right\}+\frac{c(1+c)}{2m^3(1-c)^3}\left\{\frac{l_{k-1}+l_{k+1}}{l_k}-2\right\}\\&-\frac{c}{6m^2(1-c)^2}\left\{(2+v)\frac{l_{k-1}}{l_k}-\frac{1+2v}{v}\cdot\frac{l_{k+1}}{l_k}+3-4v+v^2\right\}+\frac{c}{m(1-c)}\end{aligned}\right\}\ldots(3)$$

This result is exact, according to the suppositions we have made; but it is far too complicated to be of any practical use, in its present form, being considerably more lengthy than Professor De Morgan's formula.

In order to verify the formula, let us apply it to the case of a perpetual annuity certain; for which purpose, we put $l_{k-1}=l_k=l_{k+1}$, and $a_k=\frac{1}{i}$; then it will be found that the formula reduces to $\frac{c}{m(1-c)}=\frac{v^{\frac{1}{m}}}{m(1-v^{\frac{1}{m}})}$, which is the correct value of the annuity on the suppositions we have made.

For $\quad\frac{1}{m}\left\{v^{\frac{1}{m}}+v^{\frac{2}{m}}+v^{\frac{3}{m}}+\;\ldots\;\text{ad inf.}\right\}=\frac{1}{m}\cdot\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}.$

The above formula (3) differs from that obtained by Professor De Morgan, for the reason stated above. His formula becomes in the case of the perpetual annuity certain $\frac{1}{i}+\frac{m-1}{2m}$. In order to ascertain how far the two results differ, we must expand the former by powers of i.

Thus since $v=\dfrac{1}{1+i}$ and $v^{\frac{1}{m}}=(1+i)^{-\frac{1}{m}}$, we have

$$\frac{1}{m}\cdot\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}=\frac{1}{m}\cdot\frac{1}{v^{-\frac{1}{m}}-1}=\frac{1}{m}\cdot\frac{1}{(1+i)^{\frac{1}{m}}-1}$$

$$=\frac{1}{m}\cdot\frac{1}{\frac{1}{m}i-\frac{m-1}{2m^2}i^2+\frac{(m-1)(2m-1)}{6m^3}i^3-\&c.}$$

$$=\frac{1}{i}\cdot\frac{1}{1-\frac{m-1}{2m}i\left\{1-\frac{2m-1}{3m}i+\frac{(2m-1)(3m-1)}{12m^2}i^2-\&c.\right\}}$$

$$=\frac{1}{i}\left[1+\frac{m-1}{2m}i\left\{1-\frac{2m-1}{3m}i+\frac{(2m-1)(3m-1)}{12m^2}i^2-\ \ldots\right\}\right.$$

$$\left.+\frac{(m-1)^2}{4m^2}i^2\left\{1-\frac{2m-1}{3m}i+\ldots\right\}^2+\left(\frac{m-1}{2m}\right)^3 i^3+\ldots\right]$$

$$=\frac{1}{i}+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}i+\frac{m^2-1}{24m^2}i^2-\ \ldots \qquad \ldots\ (4)$$

In order to obtain a practical formula from the complicated one above given (3), it will be necessary to expand its terms by powers of i. Before doing this, I propose to examine the formula given by Mr. Woolhouse (*Assurance Magazine*, vol. xi., p. 327),

$$a'_k=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

where μ is the "force of mortality" or the "instantaneous rate of mortality" at the age k, and is equal to the value of $-\dfrac{1}{l_x}\dfrac{d}{dx}l_x$, when x is made equal to k; or approximately equal to $\dfrac{l_{k-1}-l_{k+1}}{2l_k}$; and δ is the "force of discount" and is equal to $-\log_e v$, or $\log_e(1+i)$. Mr. Woolhouse has arrived at this result as a simple case of a problem which he has investigated at great length. It may be readily proved by the following method, which although much shorter than Mr. Woolhouse's, is less elementary. The value of an annuity of £1 payable yearly is

$$a_k=\frac{1}{l_k}\left\{l_{k+1}v+l_{k+2}v^2+\ \ldots\ \right\}$$

and that of an annuity of £1 payable by m equal instalments in each year is

$$a'_k=\frac{1}{ml_k}\left\{l_{k+\frac{1}{m}}v^{\frac{1}{m}}+l_{k+\frac{2}{m}}v^{\frac{2}{m}}+\ \ldots\ \right\}$$

Now by a well known theorem proved in the Calculus of Finite Differences, u_x being any function of x,

$$\Sigma u_x=\int u_x dx-\frac{u_x}{2}+\frac{1}{12}\frac{du_x}{dx}-\frac{1}{720}\frac{d^3u_x}{dx^3}+\&c.$$

which between the limits a and b, which are both integers and $b>a$, becomes

$$\Sigma u_b-\Sigma u_a=\int_a^b u_x dx-\frac{u_b}{2}+\frac{1}{12}\left(\frac{du_x}{dx}\right)_b-\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_b+\&c.$$
$$+\frac{u_a}{2}-\frac{1}{12}\left(\frac{du_x}{dx}\right)_a+\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_a-\&c.$$

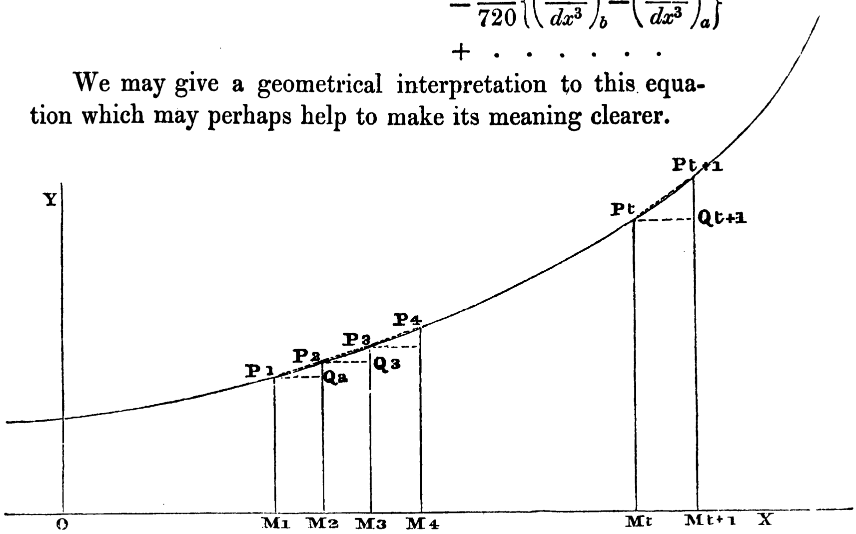
But the first member of this equation is equal to $u_a+u_{a+1}+u_{a+2}+\ldots\ldots+u_{b-1}$, so that we get

$$\left.\begin{aligned}u_a+u_{a+1}+\ldots\ldots+u_{b-1}=&\int_a^b u_x dx\\ &-\frac{u_b}{2}+\frac{1}{12}\left(\frac{du_x}{dx}\right)_b-\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_b+\ldots\ldots\\ &+\frac{u_a}{2}-\frac{1}{12}\left(\frac{du_x}{dx}\right)_a+\frac{1}{720}\left(\frac{d^3u_x}{dx^3}\right)_a-\ldots\ldots\end{aligned}\right\}\ldots(5)$$

We also have

$$\frac{u_a}{2}+u_{a+1}+\ldots\ldots+u_{b-1}+\frac{u_b}{2}=\int_a^b u_x dx+\frac{1}{12}\left\{\left(\frac{du_x}{dx}\right)_b-\left(\frac{du_x}{dx}\right)_a\right\}^*$$
$$-\frac{1}{720}\left\{\left(\frac{d^3u_x}{dx^3}\right)_b-\left(\frac{d^3u_x}{dx^3}\right)_a\right\}$$
$$+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

We may give a geometrical interpretation to this equation which may perhaps help to make its meaning clearer.



Let $OM_1=a$, $OM_{t+1}=b$, so that $OM_{t+1}-OM_1=b-a=t$, suppose; and $M_1M_2=M_2M_3=M_3M_4=\ldots\ldots=M_tM_{t+1}=1$.

Also $P_1M_1=u_a$, $P_2M_2=u_{a+1}$, $\ldots\ldots P_{t+1}M_{t+1}=u_b$; then will the curvilinear area $P_1M_1M_{t+1}P_{t+1}$ be equal to $\int_a^b u_x dx$.

* If the difference between successive values of x be h instead of 1, this formula becomes

$$\frac{u_a}{2}+u_{a+h}+u_{a+2h}+\ldots+u_{b-h}+\frac{u_b}{2}=\frac{1}{h}\int_a^b u_x dx+\frac{h}{12}(u'_b-u'_a)-\frac{h^3}{720}(u'''_b-u'''_a)+\ldots$$

where u'_a, u'''_a, &c., are put for $\left(\frac{du_x}{dx}\right)_a$, $\left(\frac{d^3u_x}{dx^3}\right)_a$, &c.

Compare the formulæ given by Mr. Woolhouse, vol. xi., pp. 317, 331.

Also the area of the rectangle P_1M_2 is equal to $P_1M_1 \times M_1M_2 = u_a \times 1 = u_a$; the area of the rectangle P_2M_3 is u_{a+1}; so that $u_a + u_{a+1} + \ldots\ldots + u_{b-1}$ is the sum of the areas of the rectangles P_1M_2, P_2M_3, ... P_tM_{t+1}.

Again the area of the triangle $P_1P_2Q_2$ is equal to $\frac{1}{2} \times P_1Q_2 \times P_2Q_2 = \frac{1}{2} \times 1 \times (u_{a+1} - u_a) = \frac{u_{a+1} - u_a}{2}$: the area of the triangle $P_2P_3Q_3$ is $\frac{u_{a+2} - u_{a+1}}{2}$: and so on, till we come to the triangle $P_tP_{t+1}Q_{t+1}$, the area of which is $\frac{u_b - u_{b-1}}{2}$; so that the sum of the areas of the triangles is $\frac{u_b - u_a}{2}$, and the area of the polygon $M_1P_1P_2P_3$ .... $P_{t+1}M_{t+1}$ is equal to $\frac{u_a}{2} + u_{a+1} + u_{a+2} + \ldots\ldots + u_{b-1} + \frac{u_b}{2}$.

This result may also be arrived at by observing that the areas of the trapeziums, $P_1M_1M_2P_2$, $P_2M_2M_3P_3$, ... $P_tM_tM_{t+1}P_{t+1}$, are $\frac{u_a + u_{a+1}}{2}$, $\frac{u_{a+1} + u_{a+2}}{2}$, ... $\frac{u_{b-1} + u_b}{2}$, respectively.

Thus we see that the sum of the areas of the curvilinear figures P_1P_2, P_2P_3, ... P_tP_{t+1}, is equal to

$$\frac{1}{12}\left\{\left(\frac{du_x}{dx}\right)_b - \left(\frac{du_x}{dx}\right)_a\right\} - \frac{1}{720}\left\{\left(\frac{d^3u_x}{dx^3}\right)_b - \left(\frac{d^3u_x}{dx^3}\right)_a\right\} + \ldots$$

Resuming the equation (5), first write $l_{k+x}v^x$ for u_x, and make $a=k$, and b equal to the extreme age in the table (z). Then the first member of the equation becomes $l_k + l_{k+1}v + l_{k+2}v^2 + \ldots\ldots$, which is equal to $l_k(1 + a_k)$. In the second member we observe that at the extremity of life, the values of u_x, $\frac{du_x}{dx}$, $\frac{d^3u_x}{dx^3}$ .... all vanish, so that we get, the limits of x being 0 and $z-k$,

$$l_k(1 + a_k) = \int_0^{z-k} l_{k+x}v^x dx + \frac{l_k}{2} - \frac{1}{12}\frac{d}{dx}(v^x l_{k+x})_0 + \frac{1}{720}\frac{d^3}{dx^3}(v^x l_{k+x})_0 - \ldots$$

$$\text{or } l_k a_k = \mathrm{P} - \frac{l_k}{2} - \frac{\mathrm{Q}}{12} + \frac{\mathrm{R}}{720} - \ldots\ldots, \text{ suppose.}$$

Next, putting $l_{k+\frac{x}{m}} \cdot v^{\frac{x}{m}}$ for u_x in the equation (5), we get similarly, the limits of x being now 0 and $m(z-k)$

$$l_k\left(\frac{1}{m} + a'_k\right) = \frac{1}{m}\int_0^{m(z-k)} l_{k+\frac{x}{m}} v^{\frac{x}{m}} dx + \frac{l_k}{2m} - \frac{1}{12m}\frac{d}{dx}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0$$

$$+ \frac{1}{720m}\frac{d^3}{dx^3}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 - \ldots\ldots$$

But
$$\int_0^{m(z-k)} l_{k+\frac{x}{m}} v^{\frac{x}{m}} dx = m\int_0^{z-k} l_{k+x} v^x dx = m\mathrm{P}$$

$$\frac{d}{dx}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 = \frac{1}{m}\frac{d}{dx}\left(l_{k+x}v^x\right)_0 = \frac{\mathrm{Q}}{m}$$

$$\frac{d^3}{dx^3}\left(l_{k+\frac{x}{m}} v^{\frac{x}{m}}\right)_0 = \frac{1}{m^3}\frac{d^3}{dx^3}\left(l_{k+x}v^x\right)_0 = \frac{\mathrm{R}}{m^3}$$

$$\&c. \quad = \quad \&c.$$

whence
$$l_k\left(\frac{1}{m} + a'_k\right) = \mathrm{P} + \frac{l_k}{2m} - \frac{\mathrm{Q}}{12m^2} + \frac{\mathrm{R}}{720m^4} - \&c.$$

and
$$l_k a'_k = \mathrm{P} - \frac{l_k}{2m} - \frac{\mathrm{Q}}{12m^2} + \frac{\mathrm{R}}{720m^4} - \&c.$$

$$l_k(a'_k - a_k) = \frac{l_k}{2}\cdot\frac{m-1}{m} + \frac{\mathrm{Q}}{12}\,\frac{m^2-1}{m^2} - \frac{\mathrm{R}}{720}\,\frac{m^4-1}{m^4} + \&c. \,.\,. \quad (6)$$

It now only remains to find the values of Q, R, &c. Thus, we have

$$\frac{d}{dx}(v^x l_{k+x}) = v^x \log_e v \, l_{k+x} + v^x \frac{d}{dx} l_{k+x}$$

and putting $x=0$ we get

$$\begin{aligned}\mathrm{Q} &= \log_e v \,.\, l_k + \left(\frac{d}{dx} l_{k+x}\right)_0 \\ &= -l_k \log_e(1+i) - \mu l_k, \text{ since } \mu = -\frac{1}{l_k}\cdot\left(\frac{d}{dx} l_{k+x}\right)_0 \\ &= -l_k(\mu+\delta)\end{aligned}$$

Also

$$\frac{d^3}{dx^3}(v^x l_{k+x}) = v^x \log^3 v l_{k+x} + 3v^x \log^2 v \frac{d}{dx} l_{k+x} + 3v^x \log v \frac{d^2}{dx^2} l_{k+x} + v^x \frac{d^3}{dx^3} l_{k+x}$$

whence, making $x=0$, as before,

$$\begin{aligned}\mathrm{R} &= \log^3 v l_k + 3\log^2 v\left(\frac{d}{dx} l_{k+x}\right)_0 + 3\log v\left(\frac{d^2}{dx^2} l_{k+x}\right)_0 + \left(\frac{d^3}{dx^3} l_{k+x}\right)_0 \\ &= -\delta^3 l_k - 3\delta^2 \mu l_k - 3\delta l''_k + l'''_k.\end{aligned}$$

Substituting in (6), and dividing by l_k, we get

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta) + \frac{m^4-1}{720m^4}\left(\delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k}\right) - \&c.$$

Here μ is approximately equal to $\dfrac{l_{k-1}-l_{k+1}}{2l_k}$. Its value, as well as those of l''_k and l'''_k, may be found more exactly as follows. Let the tabulated values of l_x adjoining l_k be differenced, as is done

by Mr. Woolhouse (*Assurance Magazine,* vol. xi. p. 64), then by the formula which he has given on p. 68, we have

$$l_{k+x}=l_k+\left(a_0-\frac{c_0}{6}+\frac{e_0}{30}-\frac{g_0}{140}\right)x$$
$$+\left(\frac{b_0}{2}-\frac{d_0}{24}+\frac{f_0}{180}-\frac{h_0}{1120}\right)x^2$$
$$+\left(\frac{c_0}{6}-\frac{e_0}{24}+\frac{7g_0}{720}\right)x^3+\&c.$$

Hence $$\left(\frac{d}{dx}l_{k+x}\right)_0=-\mu l_k=a_0-\frac{c_0}{6}+\frac{e_0}{30}-\frac{g_0}{140}$$

$$l''=\left(\frac{d^2}{dx^2}l_{k+x}\right)_0=2\left(\frac{b_0}{2}-\frac{d_0}{24}+\frac{f_0}{180}-\frac{h_0}{1120}\right)$$

$$l'''_k=\left(\frac{d^3}{dx^3}l_{k+x}\right)_0=6\left(\frac{c_0}{6}-\frac{e_0}{24}+\frac{7g_0}{720}\right)$$

In order to estimate the magnitude of the various terms in the above expression for a'_k, let us take a numerical example; and since the importance of the corrections introduced by the third and fourth terms will be greatest at advanced ages, let us suppose $k=85$. Then, using the "Experience" table, we have

Age. (x)	Number Living. (l_x)	Δ_1 ($-d_x$)	Δ_2	Δ_3	Δ_4	Δ_5
80	13290					
		−1866				
81	11424		+136			
		−1730		+12		
82	9694		148		−5	
		−1582		7		+2
83	8112		155		−3	
		−1427		4		−3
84	6685		159		−6	
		−1268		−2		+4
85	5417	. . . .	157	. . .	−2	. .
		−1111		−4		0
86	4306		153		−2	
		−958		−6		−1
87	3348		147		−3	
		−811		−9		+2
88	2537		138		−1	
		−673		−10		
89	1864		128			
		−545				
90	1319					

Here $$a_0=\tfrac{1}{2}(-1268-1111)=-\frac{2379}{2}=-1189{\cdot}5$$
$$b_0=157$$
$$c_0=\tfrac{1}{2}(-2-4)=-3$$
$$d_0=-2$$
$$e_0=\tfrac{1}{2}(4+0)=2$$

Also supposing the rate of interest to be 3 per cent. we have $\delta=\log_e(1+i)=\log_e(1{\cdot}03)={\cdot}02956.$

Hence we find

$$\mu = \cdot 21948,\ \frac{l''_k}{l_k} = \frac{157\cdot 167}{5417} = \cdot 029014,\ \frac{l'''_k}{l_k} = -\frac{3\cdot 5}{5417} = -\cdot 000646$$

$$\delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k} = \begin{array}{l} \cdot 000026 \\ +\cdot 000575 \\ +\cdot 002573 \\ +\cdot 000646 \\ \hline = \cdot 003820 \end{array}$$

and

$$\frac{1}{720}\left(\delta^3 + 3\delta^2\mu + \frac{3\delta l''_k}{l_k} - \frac{l'''_k}{l_k}\right) = \cdot 000005$$

We hence conclude that, omitting the term involving this quantity, and the following terms, the value of a'_k may be found correct to five decimal places from the equation

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu+\delta) \quad \ldots\ldots \quad (7)$$

Again

$$\mu + \delta = \cdot 21948 + \cdot 02956$$
$$= \cdot 24905$$

and

$$\frac{\mu+\delta}{12} = \cdot 02075\ ;$$

whence we conclude that the common approximation $a_k + \frac{m-1}{2m}$ is only true to the first decimal place in this instance.

Lastly, if we take the approximate value of μ, viz.—$\frac{a_0}{l_k}$ ($=\cdot 21958$), the error in the value of the annuity is equal to $-\frac{m^2-1}{12m^2} \times \cdot 000105$, and will not amount to more than a unit in the fifth decimal place. For any purpose that is ever likely to be required, we may therefore use the formula

$$a'_k = a_k + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left\{\frac{l_{k-1}-l_{k+1}}{2l_k} + \log_e(1+i)\right\}$$

For younger ages the last term of this equation will be much less important, but it will be found in all cases to affect the third place of decimals. It will also be noticed that the value of the third term becomes slightly greater as the rate of interest is greater.

The preceding expression for a'_k may be usefully compared with the one found above (4) for the value of a perpetual annuity certain. Thus, putting $l_{k-1} = l_{k+1}$, and $a_k = \frac{1}{i}$, it becomes

$$\frac{1}{i} + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left\{i - \frac{i^2}{2} + \ldots\ldots\right\}$$

which agrees with (4) as far as the term involving i^2.

Again, at 3 per cent. interest $i=\cdot03$, $\delta=\cdot02956$ $\therefore i-\delta=\cdot00044$ and $\frac{m^2-1}{12m^2}(i-\delta)<00004$, so that if we use the formula

$$a'_k=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)\quad . \; . \; . \; . \quad (8)$$

we may expect to obtain the value of a'_k correct to four places of decimals.

In this equation making $m=2, 4, \infty$, in succession, and putting μ for its approximate value $\frac{l_{k-1}-l_{k+1}}{2l_k}$, we have the value of an annuity of £1 payable half yearly equal to $a_k+\frac{1}{4}-\frac{1}{16}(\mu+i)$

quarterly „ $a_k+\frac{3}{8}-\frac{5}{64}(\mu+i)$

momently „ $a_k+\frac{1}{2}-\frac{1}{12}(\mu+i)$

Now, returning to the equation (3), our object will be to expand it in a series of ascending powers of i.

We have

$$c=v^{\frac{1}{m}}=\frac{1}{(1+i)^{\frac{1}{m}}}=(1+i)^{-\frac{1}{m}}$$

or

$$c=1-\frac{i}{m}+\frac{m+1}{2m^2}i^2-\frac{(m+1)(2m+1)}{6m^3}i^3+\frac{(m+1)(2m+1)(3m+1)}{24m^4}i^4-\text{\&c.}$$

$$\therefore \frac{1}{m(1-c)}=\frac{1}{i\left\{1-\frac{m+1}{2m}i+\frac{(m+1)(2m+1)}{6m^2}i^2-\frac{(m+1)(2m+1)(3m+1)}{24m^3}i^3+\ldots\right\}}$$

$$=\frac{1}{i}\,\frac{1}{1-\frac{m+1}{2m}i\left\{1-\frac{2m+1}{3m}i+\frac{(2m+1)(3m+1)}{12m^2}i^2\ldots\right\}}$$

Hence

$$\frac{1}{m^t(1-c)^t}=\frac{1}{i^t}\left[1-\frac{m+1}{2m}i\left\{1-\frac{2m+1}{3m}i+\frac{(2m+1)(3m+1)}{12m^2}i^2-\ldots\right\}\right]^{-t}$$

$$=\frac{1}{i^t}\left[1+t\cdot\frac{m+1}{2m}i\left\{1-\frac{2m+1}{3m}i+\frac{(2m+1)(3m+1)}{12m^2}i^2-\ldots\right\}\right.$$

$$\left.+\frac{t(t+1)}{1.2}\,\frac{(m+1)^2}{4m^2}i^2\left\{1-\frac{2m+1}{3m}i+\ldots\right\}^2+\frac{t(t+1)(t+2)}{6}\,\frac{(m+1)^3}{8m^3}i^3+\ldots\right]$$

$$=\frac{1}{i^t}\Big[1+t\frac{m+1}{2m}i+t\cdot\frac{m+1}{2m^2}i^2\left\{-\frac{2m+1}{3}+\frac{t+1}{2}\frac{m+1}{2}\right\}$$

$$+t\cdot\frac{m+1}{2m^3}i^3\left\{\frac{(2m+1)(3m+1)}{12}-\frac{(t+1)(m+1)(2m+1)}{12}\right.$$

$$\left.+\frac{(t+1)(t+2)(m+1)^2}{24}\right\}+\ldots\ldots\Big]$$

$$=\frac{1}{i^t}\left\{1+t\frac{m+1}{2m}i+t\frac{m+1}{24m^2}(\overline{3t-5}\,.m+3t-1)i^2\right.$$

$$\left.+t\frac{m+1}{48m^3}[(t^2-t+10)m^2+2(t^2+4)m+t^2+t+2]i^3+\ldots\right\}$$

Now making $t=1, 2, 3, 4$, successively

$$\frac{1}{m(1-c)}=\frac{1}{i}\left\{1+\frac{m+1}{2m}i-\frac{m^2-1}{12m^2}i^2+\frac{(m+1)(5m^2+5m+2)}{24m^3}i^3-\&c.\right\}$$

$$\frac{1}{m^2(1-c)^2}=\frac{1}{i^2}\left\{1+\frac{m+1}{m}i+\frac{(m+1)(m+5)}{12m^2}i^2\right.$$

$$\left.+\frac{(m+1)(3m^2+4m+2)}{6m^3}i^3+\ldots\right\}$$

$$\frac{1}{m^3(1-c)^3}=\frac{1}{i^3}\left\{1+\frac{3}{2}\frac{m+1}{m}i+\frac{(m+1)(m+2)}{2m^2}i^2\right.$$

$$\left.+\frac{(m+1)(16m^2+26m+15)}{16m^3}i^3+\ldots\right\}$$

$$\frac{1}{m^4(1-c)^4}=\frac{1}{i^4}\left\{1+2\frac{m+1}{m}i+\frac{(m+1)(7m+11)}{6m^2}i^2\right.$$

$$\left.+\frac{(m+1)(11m^2+20m+11)}{6m^3}i^3+\ldots\right\}$$

Also $\dfrac{c(1-v)^4}{v^2}=\dfrac{v^{\frac{1}{m}}(1-v)^4}{v^2}=\dfrac{i^4}{(1+i)^{2+\frac{1}{m}}}$

$$=i^4(1+i)^{-\frac{2m+1}{m}}$$

$$=i^4\left\{1-\frac{2m+1}{m}i+\frac{(2m+1)(3m+1)}{2m^2}i^2\right.$$

$$\left.-\frac{(2m+1)(3m+1)(4m+1)}{6m^3}i^3+\ldots\right\}$$

and $\qquad c^2=(1+i)^{-\frac{2}{m}}$

$$=1-\frac{2}{m}i+\frac{2+m}{m^2}i^2-\frac{2(2+m)(1+m)}{3m^3}i^3+\ldots$$

and $\qquad 4c=4-\dfrac{4i}{m}+\dfrac{2(m+1)}{m^2}i^2-\dfrac{2}{3}\dfrac{(m+1)(2m+1)}{m^3}i^3+\ldots$

$$\therefore 1+4c+c^2=6-\frac{6i}{m}+\frac{3m+4}{m^2}i^2-\frac{2(m+1)^2}{m^3}i^3+\ldots.$$

Again,

$$m(1-c)=i\left\{1-\frac{m+1}{2m}i+\frac{(m+1)(2m+1)}{6m^2}i^2\right.$$
$$\left.-\frac{\overline{m+1}.\overline{2m+1}.\overline{3m+1}}{24m^3}i^3\right\}$$

$$m^2(1-c)^2=i^2\left\{1-\frac{m+1}{m}i+\frac{(m+1)(11m+7)}{12m^2}i^2+\ldots.\right\}$$

$$\therefore\ 1+4c+c^2-m^2(1-c)^2=6-\frac{6i}{m}-\frac{m^2-3m-4}{m^2}i^2$$
$$+\frac{(m+1)(m^2-2m-2)}{m^3}i^3-\ldots.$$

Substituting these values, the coefficient of a_k in (3) becomes therefore

$$\left\{1-\frac{i}{m}-\frac{m^2-3m-4}{6m^2}i^2+\frac{\overline{m+1}.\overline{m^2-2m-2}}{6m^3}i^3-\ldots.\right\}$$
$$\times\left\{1-\frac{2m+1}{m}i+\frac{\overline{2m+1}.\overline{3m+1}}{2m^2}i^2-\frac{\overline{2m+1}.\overline{3m+1}.\overline{4m+1}}{6m^3}i^3+\ldots\right\}$$
$$\times\left\{1+2\times\frac{m+1}{m}i+\frac{\overline{m+1}.\overline{7m+11}}{6m^2}i^2+\frac{\overline{m+1}.\overline{11m^2+20m+11}}{6m^3}i^3+\ldots\right\}$$
$$=1\ldots\ldots\ldots+\frac{10m^3-25m^2+20m+5}{6m^3}i^3+\ldots.$$
$$=1+\frac{5}{6}\frac{(2m+1)(m+1)^2}{m^3}i^3+\ldots.$$

Thus we see that the coefficient of a_k does not contain any terms involving the first and second powers of i; and it seems a fair inference that if we found in the same way the value of the annuity on the supposition that the deaths in each year were distributed according to constant *fifth* differences, then the term involving i^3 would also disappear from the coefficient of a_k; and that further terms would disappear from the coefficient if the approximation were continued further, since in the formula obtained by the more correct method given by Mr. Woolhouse, the coefficient of a_k is simply unity.

We may obtain in the same way the values of the other terms in (3), but the following method is more simple. Put $1-c=y$, then the coefficient of $\frac{l_{k-1}}{l_k}+v$ becomes, since $c=1-y$,

$$\{-(1+4c+c^2)(c-c^{m+1})+3mc(1-c^2)-m^2(1-c)^2(2c+c^{m+1})\}\div 6m^4(1-c)^4$$

$$=\frac{c}{6m^4y^4}\left[-(6-6y+y^2)\{1-(1-y)^m\}+3m(2y-y^2)-m^2y^2\{2+(1-y)^m\}\right]$$

$$=\frac{c}{6m^4y^4}\left[-6+6(m+1)y-(2m^2+3m+1)y^2+\{6-6y-(m^2-1)y^2\}(1-y)^m\right]$$

$$=\frac{c}{6m^4y^4}\left[-6+6(m+1)y-(2m^2+3m+1)y^2+\{6-6y-(m^2-1)y^2\}\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{24}y^4\ldots\}\right]$$

$$=\frac{c}{6m^4y^4}\Big[-6+(6m+1)y-(2m^2+3m+1)y^2+6-6my+3m.\overline{m-1}.y^2-m.\overline{m-1}.\overline{m-2}.y^3+\frac{1}{4}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.y^4$$

$$-\frac{1}{20}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}.y^5\ldots\ldots+(-1)^r6\cdot\frac{m.\overline{m-1}.\overline{m-2}\ldots\ldots\overline{m-r+1}}{\underline{|r}}y^r+\ldots\ldots$$

$$-6y+6my^2-3m.\overline{m-1}y^3+m.\overline{m-1}.\overline{m-2}.y^4-\frac{1}{4}m.\overline{m-1}.\overline{m-2}.\overline{m-3}.y^5\ldots\ldots+(-1)^r6\cdot\frac{m.\overline{m-1}\ldots\ldots\overline{m-r+2}}{\underline{|r-1}}y^r+\ldots\ldots$$

$$-(m^2-1)y^2+m(m^2-1)y^3-\frac{1}{2}m.\overline{m-1}.(m^2-1)y^4+\frac{1}{6}m.\overline{m-1}.\overline{m-2}(m^2-1)y^5\ldots\ldots-(-1)^r\frac{(m^2-1)m.\overline{m-1}\ldots\ldots\overline{m-r+3}}{\underline{|r-2}}y^r+\ldots\ldots$$

$$=\frac{c}{6m^4y^4}\left[-\frac{m^2(m^2-1)}{6}y^4+\frac{m.\overline{m-1}.\overline{m-2}}{60}(7m^2+6m-1)y^5+\ldots+(-1)^r\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r}}\times(r-3)\{(r+2)m^2+6m-r-4\}y^r+\ldots\right]$$

$$=\frac{1-y}{6m^3}\left[-\frac{m(m^2-1)}{4}+\frac{\overline{m-1}.\overline{m-2}}{60}(7m^2+6m-1)y\ldots\ldots+(-1)^r\frac{\overline{m-1}.\overline{m-2}\ldots\ldots\overline{m-r+3}}{\underline{|r}}\times(r-3)\{(r+2)m^2+6m-r-4\}y^{r-4}+\ldots\right]$$

$$=\frac{1}{6m^3}\left[-\frac{m.(m^2-1)}{4}+\frac{(m^2-1)(7m^2+2)}{60}y+\ldots.\right]$$

$$=-\frac{m^2-1}{24m^2}+\frac{(m-1)(7m^2+2)}{360m^4}i,$$ approximately, since $y=1-c=1-(1+i)^{-\frac{1}{m}}=\frac{i}{m}-\frac{m+1}{2m^2}i^2+\ldots.$

The coefficient of $-\frac{l_{k+1}}{l_k}$ becomes $$\frac{c}{6m^4v(1-c)^4}\{(1+4c+c^2)(1-c^m)-3mc^m(1-c^2)-m^2(1+2c^m)(1-c)^2\}$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[(6-6y+y^2)\{1-(1-y)^m\}-3m(1-y)^m(2y-y^2)-m^2y^2\{1+2(1-y)^m\}\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[6-6y-(m^2-1)y^2-\{6-6y+y^2+6my-3my^2+2m^2y^2\}(1-y)^m\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[6-6y-(m^2-1)y^2-\{6+6(m-1)y+(2m^2-3m+1)y^2\}\left\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{24}y^4+\ldots\right\}\right]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\Big[6-6y-(m^2-1)y^2-6+6my-3m.\overline{m-1}.y^2+m.\overline{m-1}.\overline{m-2}.y^3-\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{4}.y^4+\frac{m.\overline{m-1}\ldots\overline{m-4}}{20}y^5-\ldots$$

$$+(-1)^r6\frac{m.\overline{m-1}\ldots\overline{m-r+1}}{\lfloor r}y^r+\ldots$$

$$-6(m-1)y+6m(m-1)y^2-3m(m-1)^2y^3+(m-1)m.(m-1)(m-2)y^4-\frac{m.\overline{m-1}\ldots\overline{m-3}(m-1)}{4}y^5+\ldots$$

$$-(-1)^r6(m-1)\frac{m.\overline{m-1}\ldots\overline{m-r+2}}{\lfloor r-1}y^r+\ldots$$

$$-(2m^2-3m+1)y^2+m(2m^2-3m+1)y^3-(m-1)(2m-1)\frac{m.\overline{m-1}}{2}y^4+(m-1)(2m-1)\frac{m.\overline{m-1}.\overline{m-2}}{6}y^5-\ldots.$$

$$+(-1)^r(2m^2-3m+1)\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r-2}}y^r+\ldots\Big]$$

$$=\frac{1}{6m^4c^{m-1}y^4}\left[-\frac{m^2(m^2-1)}{4}y^4+\frac{m.\overline{m-1}.\overline{m-2}}{60}(8m^2+9m+1)y^5-\ldots-(-1)^r\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\underline{|r}}(r-3)\{2m^2(r-1)+3m(r-2)+r-4\}y^r+\ldots\right]$$

$$=\frac{1}{6c^{m-1}}\left[-\frac{m^2-1}{4m^2}+\frac{\overline{m-1}.\overline{m-2}.(8m^2+9m+1)}{60m^3}y-\ldots\right]$$

$$=\frac{1}{6}(1+i)^{\frac{m-1}{m}}\Big[\qquad\qquad\Big]$$

$$=\frac{1}{6}\left[1+\frac{m-1}{m}i-\frac{m-1}{2m^2}i^2+\ldots\right]\left[-\frac{m^2-1}{4m^2}+\frac{\overline{m-1}.\overline{m-2}.(8m^2+9m+1)}{60m^4}i-\ldots.\right]$$

$$=\frac{1}{6}\left[-\frac{m^2-1}{4m^2}-\frac{(m^2-1)(7m^2+2)}{60m^4}i-\ldots.\right.$$

$$=-\frac{m^2-1}{24m^2}-\frac{(m^2-1)(7m^2+2)}{360m^4}i+\ldots.$$

The other term becomes similarly

$$\{(1+4c+c^2)(c-c^{m+1})-m(1-c^2)(2c+c^{m+1})-m^2(1-c)^2(c-2c^{m+1})+2m^3c(1-c)^3\}\div 2m^4(1-c)^4$$

$$=\frac{c}{2m^4(1-c)^4}\{(1+4c+c^2)(1-c^m)-m(1-c^2)(2+c^m)-m^2(1-c)^2(1-2c^m)+2m^3(1-c)^3\}$$

$$=\frac{c}{2m^4y^4}\{(6-6y+y^2)(1-c^m)-m(2y-y^2)(2+c^m)-m^2y^2(1-2c^m)+2m^3y^3\}$$

$$=\frac{c}{2m^4y^4}\left[6-6y+y^2-4my+2my^2-m^2y^2+2m^3y^3-(1-y)^m\{6-6y+y^2+2my-my^2-2m^2y^2\}\right]$$

$$=\frac{c}{2m^4y^4}\Big[6-(6+4m)y-(m^2-2m-1)y^2+2m^3y^3-\Big\{1-my+\frac{m.\overline{m-1}}{2}y^2-\frac{m.\overline{m-1}.\overline{m-2}}{6}y^3+\ldots\Big\}\{6+(2m-6)y-(2m^2+m-1)y^2\}\Big]$$

$$=\frac{c}{2m^4y^4}\Big[6-(4m+6)y-(m^2-2m-1)y^2+2m^3y^3-6+6my-3m.\overline{m-1}.y^2+m.\overline{m-1}.\overline{m-2}y^3-\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{4}y^4$$

$$+\frac{m.\overline{m-1}\ldots\overline{m-4}}{20}y^5+\ldots+(-1)^r6\cdot\frac{m.\overline{m-1}\ldots\overline{m-r+1}}{\lfloor\underline{r}}y^r+\ldots$$

$$-(2m-6)y+m(2m-6)y^2-m.\overline{m-1}.\overline{m-3}y^3+\frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{3}y^4-\frac{m.\overline{m-1}.\overline{m-2}.(m-3)^2}{12}y^5+\ldots$$

$$-(-1)^r(2m-6)\frac{m.\overline{m-1}\ldots\overline{m-r+2}}{\lfloor\underline{r-1}}y^r+\ldots$$

$$+(2m^2+m-1)y^2-m(2m^2+m-1)y^3+\frac{m.\overline{m-1}}{2}(2m^2+m-1)y^4-\frac{m.\overline{m-1}.\overline{m-2}(2m^2+m-1)}{6}y^5+\ldots$$

$$-(-1)^r(2m^2+m-1)\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\lfloor\underline{r-2}}y^r+\ldots\Big]$$

$$=\frac{c}{2m^4y^4}\Big[\frac{m.\overline{m-1}}{12}(13m^2+m)y^4-\frac{m.\overline{m-1}.\overline{m-2}}{60}(22m^2+m-1)y^5\ldots.$$

$$+(-1)^r\frac{m.\overline{m-1}\ldots\overline{m-r+3}}{\lfloor\underline{r}}\{2(r^2-3)m^2-(r-3)(r-6)m-(r-3)(r-4)\}y^r+\ldots.\Big]$$

$$=\frac{1-y}{2}\Big[\frac{(m-1)(13m+1)}{12m^2}-\frac{(m-1)(m-2)(22m^2+m-1)}{60m^3}y+\ldots\Big]$$

$$=\frac{(m-1)(13m+1)}{24m^2}-\frac{m^2-1}{60m^3}(11m^2+1)y+\ldots$$

$$=\frac{(m-1)(13m+1)}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}i,\text{ approximately.}$$

Now substituting these values in (3), the value of the correction becomes

$$\left\{\frac{l_{k-1}}{l_k}+1-i+i^2-i^3+\ldots\right\}\left\{-\frac{m^2-1}{24m^2}+\frac{(m^2-1)(7m^2+2)}{360m^4}i-\ldots\right\}$$
$$+\frac{l_{k+1}}{l_k}\left\{\frac{m^2-1}{24m^2}+\frac{(m^2-1)(7m^2+2)}{360m^4}i-\ldots\right\}$$
$$+\frac{(m-1)(13m+1)}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}i+\ldots$$
$$=\frac{m-1}{2m}-\frac{m^2-1}{24m^2}\cdot\frac{l_{k-1}-l_{k+1}}{l_k}$$
$$+\left\{\frac{m^2-1}{24m^2}-\frac{(m^2-1)(11m^2+1)}{60m^4}+\frac{(m^2-1)(7m^2+2)}{360m^4}\left(\frac{l_{k-1}+l_{k+1}}{l_k}+1\right)\right\}i, \text{ approximately.}$$
$$=\frac{m-1}{2m}-\frac{m^2-1}{24m^2}\cdot\frac{l_{k-1}-l_{k+1}}{l_k}+\frac{m^2-1}{12m^2}\left\{-\frac{2(11m^2+1)}{15m^2}+\frac{7m^2+2}{30m^2}\cdot\frac{l_{k-1}+l_{k+1}}{l_k}\right\}i.$$

But $\frac{l_{k-1}+l_{k+1}}{l_k}=2$, very nearly; and making this substitution, the value of the correction becomes

$$\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)$$

which agrees with the formula (8) we have found above, and is substantially the same as Mr. Woolhouse's formula (7).

The error caused by putting 2 for $\frac{l_{k-1}+l_{k+1}}{l_k}$ is equal to $\frac{l_{k-1}+l_{k+1}}{l_k}-2=\frac{l_{k-1}-l_k-(l_k-l_{k+1})}{l_k}=\frac{d_{k-1}-d_k}{l_k}$. Taking as before the Experience table of mortality, and making $k=85$, the value of this fraction is $\frac{157}{5417}$, or ·03 very nearly; and the value will be generally much less. This error has to be multiplied into $\frac{m^2-1}{12m^2}\cdot\frac{7m^2+2}{30m^2}\cdot i$, and the value of this quantity, making $m=2$, $i=\cdot03$, *i.e.* taking the case of an annuity payable half-yearly at 3 per cent. interest, is $\frac{3}{48}\cdot\frac{30}{120}\times\cdot03=\frac{\cdot03}{64}$, so that the amount of the error is nearly $\frac{\cdot0009}{64}$ or ·000014, and is therefore quite insignificant.

We have thus a very exact correspondence between the results obtained by the supposition that the deaths are distributed in each

year according to constant third differences, and by the more accurate method given by Mr. Woolhouse; and the problem of finding the value of a life annuity payable by m equal instalments may be considered as completely solved.

I purpose considering on a future occasion several allied questions—such as (1) the increase in the value of an annuity payable m times a year, when a proportionate part is paid to the day of death, (2) the value of an assurance payable at the instant of death, or at an assigned date—three or six months after death—(3) the value of a survivorship assurance, subject to the same modifications, (4) the value of a reversionary annuity which is to run from the date of the death of the life assured, the first payment being made twelve months after death instead of, as usually supposed, at the end of the year in which death occurs, (5) the value of the same annuity when payable half-yearly, quarterly, &c., or when a proportionate part is paid up to the date of death of the nominee. In these and other cases the usual formulæ are not strictly applicable to the questions which occur in practice; and it will be satisfactory, and may occasionally be useful, to know how to calculate the values more accurately.

For facility of reference I subjoin a list of the formulæ discussed in the preceding paper—the formulæ being first set down as given by the respective authors, and the modifications thereof made by myself being marked (*).

Milne, $a_k^{(m)} = a_k + \dfrac{m-1}{2m}$.

Baily, $a_k^{(2)} = \dfrac{\sqrt{v}}{4}\{2(1+i)^{\frac{1}{2}} + 2 + i\}a_k + \dfrac{\sqrt{v}}{4}$

$$= \left(1 + \frac{i^2}{16} - \frac{i^3}{16} + \ldots\right)a_k + \frac{1}{4} - \frac{i}{8} + \frac{3}{32}i^2 + \ldots \, *$$

" $a_k^{(4)} = \dfrac{v^{\frac{3}{4}}}{16}\{4(1+i)^{\frac{3}{4}} + (4+i)(1+i)^{\frac{1}{2}} + 2(2+i)(1+i)^{\frac{1}{4}} + 4 + 3i\}a_k$

$$+ \frac{v^{\frac{3}{4}}}{16}\{3(1+i)^{\frac{1}{2}} + 2(1+i)^{\frac{1}{4}} + 1\}$$

$$= \left\{1 + \frac{5}{16}i^2 - \ldots\right\}a_k + \frac{3}{8} - \frac{5}{32}i + \frac{15}{128}i^2 - \ldots \, *$$

Griffith Davies, $A^{(m)} = (P+Q)\mathbb{A} + P$,

where $P = \Sigma_x \dfrac{(m-x)m^{x-2}}{(m+i)^x}$, $Q = \left(1 + \dfrac{i}{m}\right)^m \Sigma_x \dfrac{xm^{x-2}}{(m+i)^x}$,

and $\mathbb{A}$ is the value of an annuity of £1 payable yearly, calculated at the true annual rate of interest $\left(1+\frac{i}{m}\right)^m-1=j$;

$$\text{or } \mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^*$$

$$\text{or } a_k^{(m)}=\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\ldots\right\}a_k+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}j+\ldots^*$$

In the last three formulæ, but in none of the others here given, i is the nominal rate of interest convertible m times a year, and j is the true annual interest.

$$\text{Woolhouse, } a_k^{(m)}=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

$$=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)^*$$

Errata in Part I.

Page 190, line 3 from bottom, for $\frac{3}{32}i^3$ read $\frac{3}{32}i^2$.

„ 192, „ 19, for $(m+1)^x$ read $(m+i)^x$.

„ 193, „ 16, for Σ^t read Σ_t.

„ „ throughout, for p_{k+t} read $p_{k,t}$.

„ „ „ for p_{k+t-1} read $p_{k,t-1}$.

„ „ line 20, for "value" read "amount".

„ 195, „ 6 from bottom, for $\frac{640+160i+20i^2+i^3}{1024}$ read $\frac{96+32i+3i^2}{4(4+i)^3}$.

On Annuities and Assurances on Successive Lives. *By* Thomas Weddle, F.R.A.S.*

MILNE, in his *Treatise on Annuities*, chapter vii., has discussed this subject with much clearness and elegance; yet his method of investigation does not seem to be the best possible, nor do I think that his results are given in forms well adapted to numerical applications. It may therefore be not altogether useless to consider the subject in a somewhat different manner, and so to exhibit the formulas as best to meet the wants of the computer.

Problem I.—To determine the present value (a_n) of an assurance of £1 payable on the failure of the last of n successive lives $A', A'' \ldots . A^{(n)}$.

* This paper appeared in the *Philosophical Magazine* for January, 1850, and is the one referred to by Mr. Peter Gray, at page 1, vol. ii., of this *Journal*.—Ed. *J. I. A.*